FINITE TEMPERATURE EFFECTIVE POTENTIAL FOR SPONTANEOUSLY BROKEN $\lambda\Phi^4$ THEORY.*

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ABSTRACT

We present a self-consistent calculation of the finite temperature effective potential for $\lambda\Phi^4$ theory in four dimensions using a composite operator effective action. We find that in a spontaneously broken theory not only the so-called daisy and superdaisy graphs contribute to the next-to-leading order thermal mass, but also resummed non-local diagrams are of the same order, thus altering the effective potential at small effective mass.

1. Introduction

The study of $\lambda\Phi^4$-theory at finite temperature is of great interest for a wide field of applications. A self-interacting scalar field serving as a simple model for the Higgs particle in the standard model of electroweak interactions may allow the study of symmetry changing phase transitions. In fact, the order of the electroweak phase transition plays a crucial rôle in the framework of cosmological scenarios as well as for the badly understood process of baryogenesis[1]. Despite the simplicity of the model, one may at least hope to gain some insight in the mechanism of the phase transition. Moreover, the theory is a suitable test ground for analytic non-perturbative methods, e.g. variational methods[2] as well as for lattice simulations.

High temperature symmetry restoration in a spontaneously broken theory was already noted by Kirzhnits and Linde[3] and worked out quantitatively subsequently[4]. The convenient tool for studying the behavior of the theory turns out to be the effective potential. Whereas the approximate critical temperature is already determined by the one loop potential, the order of the phase transition depends on the detailed shape of the potential requiring also the analysis of higher loop contributions, even for small coupling constant.

In particular, one finds quantizing the theory around the classical non-trivial minimum that the one loop self-energy at high temperature behaves like $m_T \sim \sqrt{\lambda}T$, independent of external momenta and the value of the classical minimum. Thus, for large T the thermal mass dominates over the tree-level mass and the minimum of the effective action becomes the trivial one. The theory exhibits two important features: non-temperature stable vacuum and effective temperature dependent mass. It was consequently proposed[2] to move the tree level mass into the interaction part of the Lagrangian and to start perturbation theory with the free Lagrangian

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$\frac{1}{2}\Phi(\Box + \Omega^2)\Phi$ including a yet undetermined mass parameter Ω. Then one calculates the effective potential and fixes the parameter by the 'principle of minimal sensitivity' $\partial V(\Phi_{min})/\partial\Omega = 0$.

A more systematic approach of a self-consistent loop expansion was developed already some time ago by Cornwall, Jackiw and Tomboulis (CJT) in their effective action formalism for composite operators[5]. The basic idea is to introduce a bilocal mass operator instead of the local mass Ω in the generating functional. Then one defines a generalized effective action as the double Legendre transform of the generating functional, which is now not only a functional of the expectation value of the field, $\phi = \langle\Phi\rangle$ but also depends on the expectation value of the time-ordered product $\mathrm{T}\Phi(x)\Phi(y)$. The principle of minimal sensitivity gets replaced by the functional condition $\delta\Gamma(\phi, G)/\delta G(x, y) = 0$ which is employed to solve for the unknown exact two-point Green function $G_0(\phi; x, y)$, and the conventional effective action can be reconstructed as $\Gamma(\phi, G_0(\phi; x, y))$. For a given action $I(\Phi)$ one finds[5]

$$\Gamma(\phi, G) = I(\phi) + \frac{1}{2}\mathrm{Tr}\ln(D_0 G^{-1}) + \frac{1}{2}\mathrm{Tr}(D^{-1}G - \mathbf{1}) + \Gamma^{(2)}(\phi, G),$$

$$D^{-1}(\phi; x, y) = \frac{\delta^2 I(\phi)}{\delta\phi(x)\delta\phi(y)},$$

where $D_0(x, y)$ is the free propagator derived from the part of the action which is quadratic in the fields. The quantity $\Gamma^{(2)}(\phi, G)$ contains all two–loop contributions and higher and has to be calculated as follows: Shift the field Φ in the classical action by ϕ. Then $I(\Phi + \phi)$ contains terms cubic and higher in Φ which define the vertices. $\Gamma^{(2)}(\phi, G)$ contains all two-particle-irreducible vacuum graphs with the given set of vertices and the propagators replaced by $G(x, y)$. The gap equation now reads

$$G^{-1}(x, y) = D^{-1}(x, y) + 2\frac{\delta\Gamma^{(2)}(\phi, G)}{\delta G(x, y)}.$$

2. $\lambda\Phi^4$ - theory

The graphs naively contributing up to order λ^2 are shown in Fig. 1. Although the non-local contribution Fig. 1b is formally of order λ^2 it contributes to order λ in the false vacuum since already the tree-level value of the expectation value of the field is $\phi \sim m/\sqrt{\lambda}$. Consequently, the 3-vertex given by $\lambda\phi$ is of order $\sqrt{\lambda}$ which means that in the case of a spontaneously broken theory the non-local contribution (b) is as important as the local one (a). We will show that this is also true in the quantized theory. Thus, the Hartree–Fock approximation involving only the local graph which corresponds to the superdaisy or hard thermal loop resummation at finite temperature fails to be consistent. We therefore include the non-local graph Fig. 1b in the self energy and the gap equation in Fourier-space reads

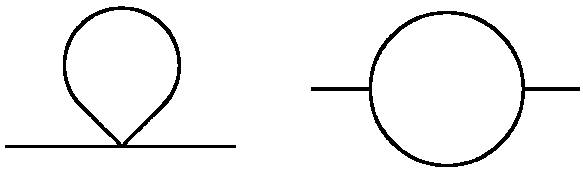

Fig. 1. Feynman graphs contributing to lowest order to the self energy. Lines denote the function $G(x,y)$, the three-vertex is given by $-\lambda\phi/3!$ and the four-vertex by $-\lambda/4!$.

$$G^{-1}(\vec{k},\omega) = \vec{k}^2 + \omega^2 + (-m^2 + \frac{\lambda}{2}\phi^2) + \frac{\lambda}{2}T\sum_{\omega'}\int\frac{d^3p}{(2\pi)^3}G(\omega',\vec{p}) + \\ +\frac{1}{2}(\lambda\phi)^2 T\sum_{\omega'}\int\frac{d^3p}{(2\pi)^3}G(\omega,\vec{p})G(\omega+\omega',\vec{p}+\vec{k}) \qquad (1)$$

where we have written down the expression in imaginary time formalism. The first term corresponding to graph (a) does not depend on external momenta and energies and thus can be treated as a mass term. A solution for the gap equation involving only the constant term can easily be found[6] and corresponds to superdaisy resummation. However, we are interested in the behavior of a broken theory, where the last term in Eq. (1) is of the same order. We employ the Ansatz

$$G^{-1}(\omega,\vec{k}) = \vec{k}^2 + \omega^2 + \mu^2 + \Pi(\omega,\vec{k})$$

and subsequently absorb all local contributions in a mass parameter μ which will be determined self-consistently afterwards. Plugging the Ansatz into Eq. (1) recasts the gap equation into an equation for $\Pi(\omega,\vec{k})$. Simple power counting now reveals that the second term in Eq. (1) diverges for large momenta $\vec{p}$. However, one may absorb the divergent part in a mass renormalization rendering the self energy $\Pi(\omega,\vec{k})$ a finite quantity.

Since we are only interested in the high temperature ($T \gg \mu$) regime of the theory one may further simplify Eq. (1) by taking only the $\omega' = 0$ mode in the second sum, since all other terms are at least suppressed by a factor $1/T$. This in turn means that it is also only the zero mode of the self energy, that contributes most. Putting $\Pi(k) = \Pi(\omega = 0, |\vec{k}|)$ we end up with the integral equation

$$\Pi(k) = \\ \frac{T}{2}(\lambda\phi)^2\int_0^\infty\frac{dpp^2}{(2\pi)^2}\int_{-1}^{1}\frac{dz}{(p^2+\mu^2+\Pi(p))(p^2+k^2+2pkz+\mu^2+\Pi(\sqrt{p^2+k^2+2pkz}))} \qquad (2)$$

which we are now going to solve.

3. Solution of the gap equation and effective potential

First, from Eq. (2) one immediately reads off the high momentum behavior $\Pi(k \to \infty) = 0$. Secondly, we note that due to the non-local character of the integral equation we have to know the function $\Pi(k)$ for all momenta, even if we were only interested in the low momentum behavior of the self energy. However, numerical study of this equation reveals that $\Pi(k)$ can be approximated by

$$\Pi(k) = \mu^2 \frac{A(\kappa)^2}{a(\kappa)^2 + (k/\mu)^2}, \quad \kappa = \frac{T(\lambda\phi)^2}{8\pi^2\mu^3} \tag{3}$$

where we introduced the dimensionless quantity κ. Since $\Pi(k)$ is of relevance in the denominator of Eq. (2) only in the infrared region $k \ll \mu$, we choose $A(\kappa)$ and $a(\kappa)$ such that Eq. (2) is fulfilled exactly for $\Pi(0)$ and $\Pi''(0)$. $A(\kappa)$ and $a(\kappa)$ are then determined by two coupled complicated transcendental equations[7]. We note that the Ansatz (3) may be interpreted as Yukawa-like potential with screening mass μa and strength $\sim A^2$

Putting all together one finally gets the gap equation for the mass parameter μ,

$$-m^2 + \frac{\lambda}{2}\phi^2 + \frac{\lambda}{2}\left(\frac{T^2}{12} + \frac{T\mu}{8\pi} f(\kappa)\right) = \mu^2 \tag{4}$$

where we have absorbed the divergencies in redefined quantities m, λ and $f = f(A(\kappa), a(\kappa))$. We find $f(0) = -2$ which corresponds to the value if one only had included the local contribution of graph (a). On the other hand for large values of the parameter κ, which in the non-trivial vacuum $\phi \neq 0$ means small mass parameter μ, one finds the asymptotic behavior $f(\kappa) \sim C\kappa^{\frac{1}{3}}$ with $C \sim -1.895\ldots$. Remarkably enough, the mass parameter cancels out in the next to leading order contribution in Eq. (4) and we are left with a term $T\mu f(\kappa) \sim T^{\frac{4}{3}}(\lambda\phi)^{\frac{2}{3}}$. In view of the result, this quite surprising behavior can already estimated from Eq. (2). Rescaling by μ such that the self–energy becomes a dimensionless, the equation may roughly be read as $\Pi^3 \sim \kappa$.

We thus find that the non-local term included in this investigation significantly alters the gap equation (4) at vanishing mass μ, that means when perturbation theory becomes problematic and infrared effects enter the game.

Since, to lowest order, the effective mass is just the second derivative of the effective potential with respect to ϕ, we are now able to reconstruct the lowest order effective potential just by twice integrating the gap equation over to ϕ. Keeping only the asymptotic form of f in the gap equation, different temperature regions may distinguished by their limiting values $T_i = m\sqrt{\frac{24}{\lambda}}(1 - F_i\lambda)^{-\frac{1}{2}}$, where $F_i = (0, \frac{216}{20\sqrt{10}}|C|^{3/2}, \frac{24\sqrt{3}}{5\sqrt{5}}|C|^{3/2}, 24 \cdot 3^{-3/2}|C|^{3/2})$. The potential has one non-trivial minimum for $T < T_0$ and develops another for $T_0 < T < T_1$. At $T = T_1$ the trivial and the non-trivial minimum have equal depth. As temperature is further raised until $T = T_2$, the non-trivial minimum vanishes completely. There exists however still a value $\phi \neq 0$, where the bending of the potential changes sign as long as the

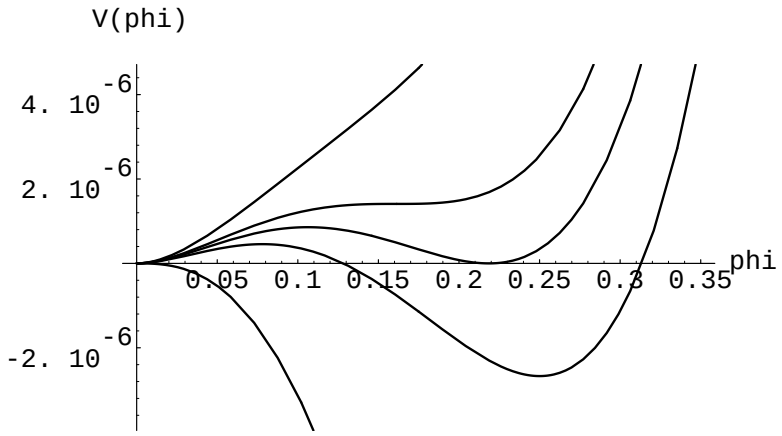

Fig. 2. The finite temperature effective potential for $\lambda = 0.1$ in units of the redefined mass m. The upper three curves correspond to the temperatures T_3, T_2, T_1 respectively, the lowest curve to the temperature T_0.

temperature is lower than T_3. The potential for $\lambda = 0.1$ is drawn in Fig. 2 for the corresponding values of the limiting temperatures where in addition, we added a curve for a temperature between $T_0 < T < T_1$.

4. Conclusion

We have shown, to lowest order however, that the naively expected resummed effective potential in a spontaneously broken theory fails to be consistent. Instead we encountered an additional contribution to the effective mass in the gap equation behaving like $\phi^{2/3}$ which cannot be found in a perturbative calculation. This may point towards a breakdown of the resummed perturbative calculation thus encouraging the extensive study of alternative methods. Nevertheless, we find that the phase transition is still of first order. A refinement of the present approach is currently under investigation.

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